

# DERIVED CATEGORY INVARIANTS AND L-SERIES

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**ABSTRACT.** We relate invariants in derived categories associated to tame actions of finite groups on projective varieties over a finite field to zeros of L-functions

## 1. INTRODUCTION

A recurring theme in the study of values of L-functions of arithmetic schemes is that these should be related to Euler characteristics of various kinds. Behind the cohomology groups needed to define such Euler characteristics are hypercohomology complexes in derived categories. In this paper we consider how to determine the additional information contained in such complexes beyond what is seen by Euler characteristics. In the geometric situations we consider, this additional information takes the form of extension classes. Our main result is that two natural extension classes constructed from étale and coherent cohomology differ by a numerical invariant which is the reciprocal of the zero of an  $L$ -function. This suggests that it may be fruitful to study relationships between derived category invariants and L-functions in more general contexts, e.g. for projective schemes over  $\mathbb{Z}$ .

We will now describe the contents of this paper. In Section 2 we consider the following geometric situation. Let  $X$  be a smooth projective variety over the algebraic closure  $\bar{k}$  of a finite field  $k$  having a tame, generically free action over  $\bar{k}$  of a finite group  $G$ . We suppose that the cohomology groups of  $\mathcal{O}_X$  vanish except in dimensions 0 and  $n$  for some integer  $n > 0$  and that the characteristic  $p$  of  $k$  divides  $\#G$ . In this case, the isomorphism class of the hypercohomology complex  $H^\bullet(X, \mathcal{O}_X)$  in the derived category of the homotopy category of  $\bar{k}[G]$ -modules is determined by its Euler characteristic and an extension class  $\beta(X, G)$  in the one-dimensional  $\bar{k}$ -vector space  $\text{Ext}_{\bar{k}[G]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X))$ . Similarly, the isomorphism class of  $H_{et}^\bullet(X, k)$  in the derived category of the homotopy category of complexes of  $k[G]$ -modules is determined by its Euler characteristic together with an extension class  $\gamma(X, G)$  in the one-dimensional  $k$ -vector space  $\text{Ext}_{k[G]}^{n+1}(H_{et}^n(X, k), H_{et}^0(X, k))$ .

In Theorem 2.9 we show that

$$\beta(X, G) = 1 \otimes \gamma(X, G)$$

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relative to a natural isomorphism

$$\bar{k} \otimes_k \mathrm{Ext}_{\bar{k}[G]}^{n+1}(H_{et}^n(X, k), H_{et}^0(X, k)) = \mathrm{Ext}_{\bar{k}[G]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X))$$

induced by the map  $k \rightarrow \mathbb{G}_a$  of étale sheaves on  $X$ . One can thus think of

$$k \cdot \beta(X, G) = k \cdot \gamma(X, G)$$

as the “étale  $k$ -line” inside  $\mathrm{Ext}_{\bar{k}[G]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X))$ .

In Section 3 we make some additional hypotheses on  $G$  and  $X$  described in Hypothesis 3.1. We assume in particular that the  $p$ -Sylow subgroups of  $G$  are cyclic and non trivial. We let  $C$  be a  $p$ -Sylow subgroup of  $G$ . Under these hypotheses we define a  $k$ -line

$$k \cdot \alpha(X, G)$$

inside  $\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X))$  which is associated to a model  $Y_0$  over  $k$  of the quotient  $Y = X/G$  of  $X$  by the action of  $G$ . One can think of  $k \cdot \alpha(X, G)$  as a  $k$ -line determined by coordinates for the one-dimensional  $\bar{k}$ -vector space  $\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X))$  which arises from the model  $Y_0$ .

The restriction map induces an isomorphism of  $\bar{k}$ -vector spaces

$$\mathrm{Ext}_{\bar{k}[G]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X)) \rightarrow \mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X)).$$

We consider  $k \cdot \beta(X, G)$  as a  $k$ -line of  $\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X))$  via this isomorphism. Our goal is to compare in  $\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(X, \mathcal{O}_X), H^0(X, \mathcal{O}_X))$  the étale  $k$ -line with the  $k$ -line provided by the model  $Y_0$ . To be more precise our main result, Theorem 3.6 and its corollaries, is that

$$k \cdot \alpha(X, G) = \zeta \cdot k \cdot \beta(X, G)$$

for a constant  $\zeta \in \bar{k}^*$  such that

$$\mu(X, G) = \zeta^{1-\#k} \in k^*$$

is independent of all choices and is the reciprocal of a zero of an  $L$ -function associated to  $X$ . If  $n$  is odd, this  $L$ -function is the numerator of the mod  $p$  zeta function of  $Y_0$  over  $k$ . If  $n$  is even, the  $L$ -function is the denominator of the mod  $p$  zeta function of the variety  $X_0$  which is the quotient of  $X$  by the group generated by a lift to  $X$  of the arithmetic Frobenius in  $\mathrm{Gal}(Y/Y_0)$ . If  $X$  is an elliptic curve and  $k = \mathbb{Z}/p$  then  $\mu(X, G)$  is simply the Hasse invariant associated to  $Y_0$  (c.f. Example 3.11).

In the last section of this paper we provide examples of projective varieties of arbitrary large dimension, endowed with an action of a cyclic group of order  $p$  for which the hypotheses of Theorem 3.6 are fulfilled.

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## 2. VARIETIES WITH TWO NON-VANISHING COHOMOLOGY GROUPS

Let  $k$  be a finite field of order  $q = p^f$ , where  $p$  is a prime, and let  $\overline{k}$  be an algebraic closure of  $k$ . We will suppose that  $G$  is a finite group of order divisible by  $p$  acting tamely and generically freely over  $\overline{k}$  on a smooth projective variety  $X$  over  $\overline{k}$  of dimension  $d$ . Let  $\pi : X \rightarrow Y = X/G$  be the quotient morphism. If  $\mathcal{F}$  is a coherent  $G$ -sheaf on  $X$ , we denote  $H^i(X, \mathcal{F})$  by  $H^i(\mathcal{F})$ .

**Hypothesis 2.1.** *There is an integer  $n \geq 1$  such that  $H^i(\mathcal{O}_X) \neq \{0\}$  if and only if  $i \in \{0, n\}$ .*

**Lemma 2.2.** *The coherent hypercohomology complex  $H^\bullet(\mathcal{O}_X)$  is isomorphic in the derived category  $D(\overline{k}G)$  of the homotopy category of complexes of  $\overline{k}[G]$ -modules to a perfect complex  $P^\bullet$  of  $\overline{k}[G]$ -modules which has trivial terms outside degrees in the interval  $[0, n]$ . This complex defines an exact sequence*

$$(2.1) \quad 0 \rightarrow H^0(\mathcal{O}_X) \rightarrow P_0 \rightarrow \cdots \rightarrow P_n \rightarrow H^n(\mathcal{O}_X) \rightarrow 0$$

and thereby an extension class  $\beta(X, G)$  in  $\mathrm{Ext}_{\overline{k}[G]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$ .

*Proof.* By a result of Nakajima [7],  $H^\bullet(\mathcal{O}_X)$  is isomorphic to a perfect complex in  $D(\overline{k}G)$  because the action of  $G$  on  $X$  is tame. Because of hypothesis 2.1, we can truncate this complex to arrive at  $P^\bullet$ .  $\square$

**Definition 2.3.** *Let  $F : \overline{k} \rightarrow \overline{k}$  be the arithmetic Frobenius automorphism over  $\overline{k}$ , so that  $F(\alpha) = \alpha^q$  for  $\alpha \in \overline{k}$ . A  $k$ -linear map  $T : M_1 \rightarrow M_2$  between vector spaces over  $\overline{k}$  will be called semilinear (resp. anti-semilinear) if  $T(\alpha \cdot m_1) = F(\alpha)T(m_1)$  (resp.  $T(\alpha \cdot m_1) = F^{-1}(\alpha)T(m_1)$ ) for  $\alpha \in \overline{k}$  and  $m_1 \in M_1$ .*

**Lemma 2.4.** *Suppose that  $\ell$  is a field of characteristic  $p$ , and that there is an exact sequence of  $\ell G$ -modules*

$$(2.2) \quad 0 \rightarrow \ell \rightarrow P_0 \rightarrow \cdots \rightarrow P_n \rightarrow M \rightarrow 0$$

in which  $P_i$  is projective and finitely generated for all  $i$ . Then  $\mathrm{Ext}_{\ell[G]}^{n+1}(M, \ell)$  is a one-dimensional  $\ell$  vector space with respect to the multiplication action of  $\ell$  on  $\ell$ . The degeneration of the spectral sequence  $H^p(G, \mathrm{Ext}_\ell^q(M, \ell)) \rightarrow \mathrm{Ext}_{\ell[G]}^{p+q}(M, \ell)$  gives an isomorphism

$$(2.3) \quad \mathrm{Ext}_{\ell[G]}^{n+1}(M, \ell) = H^{n+1}(G, \mathrm{Hom}_\ell(M, \ell))$$

*Proof.* By dimension shifting via the sequence (2.2), we get an exact sequence

$$\mathrm{Hom}_{\ell[G]}(P_0, \ell) \rightarrow \mathrm{Hom}_{\ell[G]}(\ell, \ell) \rightarrow \mathrm{Ext}_{\ell[G]}^{n+1}(M, \ell) \rightarrow 0.$$

Here  $\text{Hom}_{\ell[G]}(\ell, \ell) = \ell$ . So either  $\text{Ext}_{\ell[G]}^{n+1}(M, \ell)$  is a one-dimensional  $\ell$ -vector space, or the injection  $\ell \rightarrow P_0$  splits. However, we have assumed  $p$  divides the order of  $G$ , so  $\ell$  is not a projective  $\ell[G]$ -module and the latter alternative is impossible.  $\square$

**Corollary 2.5.** *Suppose  $\ell = \bar{k}$  and that  $T : M \rightarrow M$  is a semilinear map commuting with the action of  $G$  on  $M$ . There is a  $G$ -equivariant anti-semilinear endomorphism  $T^{-1}$  of  $\text{Hom}_{\bar{k}}(M, \bar{k})$  defined by*

$$T^{-1}(f)(m) = F^{-1}(f(T(m))) \quad \text{for } f \in \text{Hom}_{\bar{k}}(M, \bar{k}) \quad \text{and } m \in M.$$

Via (2.3) this gives an anti-semilinear action of  $T^{-1}$  on  $\text{Ext}_{\bar{k}[G]}^{n+1}(M, \bar{k}) \cong \bar{k}$ .

The following result is Lemma III.4.13 of [5]; see also [1, §XXII.1] and [6, p. 143].

**Lemma 2.6.** *Let  $V$  be a finite dimensional vector space over  $\bar{k}$ , and let  $\phi : V \rightarrow V$  be a semilinear map. Then  $V$  decomposes as a direct sum  $V = V_s \oplus V_\eta$ , where  $V_s$  and  $V_\eta$  are subspaces stable under  $\phi$ ,  $\phi$  is bijective on  $V_s$  and  $\phi$  is nilpotent on  $V_\eta$ . Moreover,  $V_s$  has a basis  $\{e_1, \dots, e_t\}$  such that  $\phi(e_i) = e_i$  for all  $i$ . It follows that  $V^\phi$  is the  $k$ -vector space having basis  $\{e_1, \dots, e_t\}$  and  $\phi - 1 : V \rightarrow V$  is surjective.*

We have an exact Artin-Schreier sequence of étale sheaves on  $X$  given by

$$(2.4) \quad 0 \longrightarrow k \longrightarrow \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \rightarrow 0$$

where  $F : \mathbb{G}_a \rightarrow \mathbb{G}_a$  is the arithmetic Frobenius morphism defined by  $\alpha \mapsto \alpha^q = F(\alpha)$  for  $\alpha$  a local section of  $\mathbb{G}_a$ . By [5, Remark III.3.8], there is an isomorphism  $H^\bullet(X, \mathcal{O}_X) \rightarrow H_{et}^\bullet(X, \mathbb{G}_a)$  in the derived category, giving isomorphisms  $H^j(\mathcal{O}_X) \rightarrow H_{et}^j(X, \mathbb{G}_a)$  for all  $j$ .

**Lemma 2.7.** *The long exact cohomology sequence associated to (2.4) splits into short exact sequences*

$$(2.5) \quad 0 \longrightarrow H_{et}^i(X, k) \longrightarrow H^i(\mathcal{O}_X) \xrightarrow{F-1} H^i(\mathcal{O}_X) \longrightarrow 0$$

for all  $i$ . The terms of this sequence are trivial if  $i \notin \{0, n\}$ . When  $i = 0$ , one has  $H_{et}^0(X, k) = k$  and  $H^0(\mathcal{O}_X) = \bar{k}$ . When  $i = n$ , there is a  $\bar{k}G$ -module isomorphism

$$(2.6) \quad H^n(\mathcal{O}_X) = H^n(\mathcal{O}_X)_{F,s} \oplus H^n(\mathcal{O}_X)_{F,\eta}$$

arising from Lemma 2.6 in which  $F$  is an isomorphism on  $H^i(\mathcal{O}_X)_{F,s}$  and nilpotent on  $H^i(\mathcal{O}_X)_{F,\eta}$ . The sequence (2.5) with  $i = n$  shows  $H^n(\mathcal{O}_X)^F = H_{et}^n(X, k)$  and  $H^n(\mathcal{O}_X)_{F,s} = \bar{k} \otimes_k H_{et}^n(X, k)$ .

*Proof.* The action of  $F$  on  $H_{et}^i(X, \mathbb{G}_a) = H^i(\mathcal{O}_X)$  is semilinear for all  $i$ . The split exact sequences (2.5) arise from the fact that by Lemma 2.6,

$$F - 1 : H^i(\mathcal{O}_X) \rightarrow H^i(\mathcal{O}_X)$$

is surjective for all  $i$ . When  $i = n$ , the decomposition in (2.6) is a  $\overline{k}G$ -module decomposition because  $F$  commutes with the action of  $G$ ,

$$H^n(\mathcal{O}_X)_{F,s} = \cap_{m \geq 1} F^m(H^n(\mathcal{O}_X))$$

and

$$H^n(\mathcal{O}_X)_{F,\eta} = \text{Kernel}(F^m : H^n(\mathcal{O}_X) \rightarrow H^n(\mathcal{O}_X)) \quad \text{if } m >> 0.$$

The sequence (2.5) shows  $H^n(\mathcal{O}_X)^F = H_{et}^n(X, k)$  so  $H^n(\mathcal{O}_X)_{F,s} = \overline{k} \otimes_k H_{et}^n(X, k)$  by Lemma 2.6  $\square$

**Lemma 2.8.** *The complex  $H_{et}^\bullet(X, k)$  is perfect, and  $H_{et}^j(X, k) \neq 0$  if and only if  $j \in \{0, n\}$ . The sequence (2.4) gives rise to a morphism*

$$(2.7) \quad H_{et}^\bullet(X, k) \rightarrow H_{et}^\bullet(X, \mathbb{G}_a) = H^\bullet(\mathcal{O}_X)$$

in the derived category of complexes of  $k[G]$ -modules. Let  $H^\bullet(\mathcal{O}_X)'$  be the complex resulting from  $H^\bullet(\mathcal{O}_X)$  by truncating  $H^\bullet(\mathcal{O}_X)$  in dimensions greater than  $n$  and by replacing  $H^n(\mathcal{O}_X)$  by the submodule  $H^n(\mathcal{O}_X)_{F,s}$  appearing in (2.6). The morphism (2.7) gives a quasi-isomorphism

$$(2.8) \quad \overline{k} \otimes_{L,k} H_{et}^\bullet(X, k) = H^\bullet(\mathcal{O}_X)'$$

of perfect complexes of  $\overline{k}[G]$ -modules, where  $L$  on the left is the left derived tensor product. The  $\overline{k}[G]$ -module  $H^n(\mathcal{O}_X)_{F,\eta}$  in (2.6) is projective.

*Proof.* Since  $X \rightarrow Y$  is a tame  $G$ -cover, the sheaf  $\pi_*k$  in the étale topology on  $Y$  is a sheaf of projective  $k[G]$ -modules. The argument of [5, Theorem VI.13.11] now shows that  $H_{et}^\bullet(X, k)$  is a perfect complex of  $k[G]$ -modules. The isomorphism (2.8) in the derived category results from the calculation of the cohomology groups  $H_{et}^i(X, k)$  in Lemma 2.7, where the left derived tensor product  $\overline{k} \otimes_{L,k}$  is just the tensor product because all  $k$ -modules are free. This implies  $H^\bullet(\mathcal{O}_X)'$  is perfect because  $H_{et}^\bullet(X, k)$  is. Because  $H^\bullet(\mathcal{O}_X)$  is also perfect, we conclude that  $H^n(\mathcal{O}_X)_{F,\eta}$  must be projective because this was the module truncated from  $H^\bullet(\mathcal{O}_X)$  in degree  $n$  in the construction of  $H^\bullet(\mathcal{O}_X)'$ .  $\square$

It follows from Lemma 2.8 that  $H_{et}^\bullet(X, k)$  is a perfect complex such that  $H_{et}^i(X, k) \neq \{0\}$  if and only if  $i \in \{0, n\}$ . Following the lines of Lemmas 2.2 and 2.4 we conclude that we can attach to  $H_{et}^\bullet(X, k)$  an extension class  $\gamma(X, G)$  in  $\text{Ext}_{k[G]}^{n+1}(H_{et}^n(X, k), H_{et}^0(X, k))$  and prove that this  $k$ -vector space is of dimension 1.

**Theorem 2.9.** *The morphism (2.7) leads to an isomorphism of one-dimensional  $\overline{k}$  vector spaces*

$$(2.9) \quad \overline{k} \otimes_k \text{Ext}_{k[G]}^{n+1}(H_{et}^n(X, k), H_{et}^0(X, k)) = \text{Ext}_{\overline{k}[G]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$$

such that

$$(2.10) \quad \beta(X, G) = 1 \otimes \gamma(X, G).$$

The action of  $F$  on  $H^n(\mathcal{O}_X)$  and on  $H^0(\mathcal{O}_X) = \bar{k}$  leads to an anti-semilinear action of  $F^{-1}$  on  $\text{Ext}_{\bar{k}[G]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$ . Via (2.9), the one-dimensional  $k$ -vector space  $L_1 = 1 \otimes_k \text{Ext}_{\bar{k}[G]}^{n+1}(H_{et}^n(X, k), H_{et}^0(X, k))$  is the subspace of  $\text{Ext}_{\bar{k}[G]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$  which is fixed by  $F^{-1}$ . In particular,  $\beta(X, G)$  is fixed by  $F^{-1}$ .

*Proof.* Since  $H^n(\mathcal{O}_X)_{F,\eta}$  is a projective  $\bar{k}[G]$ -module by Lemma 2.8, inclusion  $H^n(\mathcal{O}_X)_{F,s} \rightarrow H^n(\mathcal{O}_X)$  induces an isomorphism of one-dimensional  $\bar{k}$ -vector spaces

$$(2.11) \quad \text{Ext}_{\bar{k}[G]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X)) \rightarrow \text{Ext}_{\bar{k}[G]}^{n+1}(H^n(\mathcal{O}_X)_{F,s}, H^0(\mathcal{O}_X))$$

which sends the extension class  $\beta(X, G)$  associated to  $H^\bullet(\mathcal{O}_X)$  to the extension class  $\beta(X, G)'$  associated to  $H^\bullet(\mathcal{O}_X)'$ . In view of Lemma 2.8, the isomorphism

$$\bar{k} \otimes_{L,k} H_{et}^\bullet(X, k) \cong H^\bullet(\mathcal{O}_X)'$$

in the derived category gives an isomorphism

$$(2.12) \quad \bar{k} \otimes_k \text{Ext}_{\bar{k}[G]}^{n+1}(H_{et}^n(X, k), H_{et}^0(X, k)) = \text{Ext}_{\bar{k}[G]}^{n+1}(H^n(\mathcal{O}_X)_{F,s}, H^0(\mathcal{O}_X))$$

of one-dimensional vector spaces over  $\bar{k}$  which identifies  $1 \otimes \gamma(X, G)$  with  $\beta(X, G)'$  when  $\gamma(X, G)$  is the extension class in  $\text{Ext}_{\bar{k}[G]}^{n+1}(H_{et}^n(X, k), H_{et}^0(X, k))$  associated to  $H_{et}^\bullet(X, k)$ . Combining (2.11) and (2.12) thus leads to an isomorphism (2.9) which identifies  $\beta(X, G)$  with  $1 \otimes \gamma(X, G)$ .

The action of  $F$  on  $H^n(\mathcal{O}_X)$  is via the action of  $F$  on  $\mathcal{O}_X$  and commutes with the action of  $G$ . (This  $F$  is different from the  $\bar{k}$ -linear relative Frobenius automorphism  $F_{X/\bar{k}}$  of  $H^n(\mathcal{O}_X) = H_{et}^n(X, \mathbb{G}_a)$  described by Milne in [5, §VI.13].) Since  $F$  acts semi-linearly and fixes both  $H_{et}^n(X, k) \subset H^n(\mathcal{O}_X)$  and  $H_{et}^0(X, k) = k \subset \bar{k} = H^0(\mathcal{O}_X)$ , the remaining assertions in Theorem 2.9 follow from (2.9).  $\square$

### 3. EXTENSION CLASS INVARIANTS ARISING FROM MODELS

In this section we will assume the following strengthening of Hypothesis 2.1. .

**Hypothesis 3.1.** *The  $p$ -Sylow subgroups of the group  $G$  are cyclic and non trivial and the  $\bar{k}$ -vector spaces  $H^n(G, \bar{k})$  and  $H^{n+1}(G, \bar{k})$  are of dimension one. The variety  $X$  is of dimension  $n$  and  $H^i(\mathcal{O}_X) = \{0\}$  if and only if  $i \notin \{0, n\}$ . There exists a smooth projective variety  $Y_0$  over  $k$  for which the following is true.*

- a.  $Y = X/G = \bar{k} \otimes_k Y_0$ .
- b. *The morphism  $\tilde{\pi} : X \rightarrow Y_0$  which is the composition of  $\pi : X \rightarrow Y$  with the projection  $Y \rightarrow Y_0$  is Galois.*

We fix once for all a  $p$ -Sylow subgroup  $C$  of  $G$ . Since  $\bar{k}$  is of characteristic  $p$ , for any integer  $m$ , the restriction map induces an injection

$$(3.1) \quad \text{Res}_C^G : H^m(G, \bar{k}) \hookrightarrow H^m(C, \bar{k}).$$

Since we have assumed  $C$  to be cyclic and non-trivial, the  $\bar{k}$ -vector spaces  $H^m(C, \bar{k})$  are of dimension 1 for all  $m$ . Therefore, Hypothesis 3.1 requires that (3.1) is an isomorphism for  $m \in \{n, n+1\}$ .

**Example 3.2.** Suppose that  $G$  is the semi-direct product of a normal subgroup  $H$  of order prime to  $p$  with a non-trivial cyclic  $p$ -group  $C$ . Then the groups  $H^i(H, \bar{k})$  are trivial for  $i \geq 0$ . Therefore the inflation homomorphisms

$$\text{Inf} : H^i(G/H, \bar{k}) \rightarrow H^i(G, \bar{k})$$

are isomorphisms and

$$H^i(C, \bar{k}) \simeq H^i(G, \bar{k}), \text{ for } i \geq 0.$$

We conclude that, in this case, the dimensions of  $H^n(G, \bar{k})$  and  $H^{n+1}(G, \bar{k})$  are both equal to one as required in Hypothesis 3.1.

The aim of this section is to show that the model  $Y_0$  for  $Y$  over  $k$  leads to a class

$$\alpha(X, G) \in \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$$

which is well defined up to multiplication by an element of  $k^*$ , and which is different in general from the class obtained by restriction from the class  $\beta(X, G)$  constructed in the previous section. This new class should be understood as an obstruction to a descent problem, and to be more precise, the descent of the action  $X \times G \rightarrow X$  defined over  $\bar{k}$  to an action  $X_0 \times G \rightarrow X_0$  defined over  $k$ . The key to constructing  $\alpha(X, G)$  is given by the following three results.

**Proposition 3.3.** Let  $\Gamma = \text{Gal}(X/Y_0)$ . The morphism of sheaves on  $Y_0$  given by  $\mathcal{O}_{Y_0} \rightarrow (\tilde{\pi})_* \mathcal{O}_X$  leads to a homomorphism

$$H^n(Y_0, \mathcal{O}_{Y_0}) \rightarrow H^n(Y_0, (\tilde{\pi})_* \mathcal{O}_X) = H^n(\mathcal{O}_X)$$

and an exact sequence

$$(3.2) \quad 0 \longrightarrow W \longrightarrow H^n(Y_0, \mathcal{O}_{Y_0}) \longrightarrow H^n(\mathcal{O}_X)^\Gamma \longrightarrow W' \longrightarrow 0$$

in which  $W$  and  $W'$  are  $k$  vector spaces of dimension 1 with a trivial action of  $G$ . Tensoring  $\mathcal{O}_{Y_0} \rightarrow (\tilde{\pi})_* \mathcal{O}_X$  with  $\bar{k}$  over  $k$  gives the natural homomorphism  $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$  of sheaves on  $Y$ . Tensoring (3.2) with  $\bar{k}$  over  $k$  gives the exact sequence

$$(3.3) \quad 0 \longrightarrow \bar{k} \otimes_k W \longrightarrow H^n(Y, \mathcal{O}_Y) \longrightarrow H^n(\mathcal{O}_X)^G \longrightarrow \bar{k} \otimes_k W' \longrightarrow 0.$$

associated to the homomorphism  $H^n(Y, \mathcal{O}_Y) \rightarrow H^n(Y, \pi_* \mathcal{O}_X) = H^n(\mathcal{O}_X)$  which results from  $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ .

**Proposition 3.4.** *Suppose that  $n$  is odd. Then the trace map  $Tr_* : H^n(\mathcal{O}_X) \rightarrow H^n(\mathcal{O}_Y)$  and the inclusion  $\bar{k} \otimes_k W \rightarrow H^n(\mathcal{O}_Y)$  induce  $\bar{k}$ -linear maps*

$$\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_Y), \bar{k}) \rightarrow \mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k})$$

and

$$\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_Y), \bar{k}) \rightarrow \mathrm{Ext}_{\bar{k}[C]}^{n+1}(\bar{k} \otimes_k W, \bar{k})$$

respectively. These maps are both surjective with the same kernel, giving an isomorphism

$$(3.4) \quad \mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k}) \rightarrow \mathrm{Ext}_{\bar{k}[C]}^{n+1}(\bar{k} \otimes_k W, \bar{k}) = \bar{k} \otimes_k \mathrm{Ext}_{k[C]}^{n+1}(W, k).$$

**Proposition 3.5.** *Suppose  $n$  is even. Then the inclusion  $H^n(\mathcal{O}_X)^G \rightarrow H^n(\mathcal{O}_X)$  and the surjection  $H^n(\mathcal{O}_X)^G \rightarrow \bar{k} \otimes_k W'$  coming from (3.3) lead to  $\bar{k}$ -vector space maps*

$$\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X)) \rightarrow \mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X)^G, H^0(\mathcal{O}_X))$$

and

$$\mathrm{Ext}_{\bar{k}[C]}^{n+1}(\bar{k} \otimes_k W', H^0(\mathcal{O}_X)) \rightarrow \mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X)^G, H^0(\mathcal{O}_X))$$

respectively. These maps are both injective with the same image, leading to an isomorphism

$$(3.5) \quad \mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X)) = \mathrm{Ext}_{\bar{k}[C]}^{n+1}(\bar{k} \otimes_k W', \bar{k}) = \bar{k} \otimes_k \mathrm{Ext}_{k[C]}^{n+1}(W', k).$$

We will give the proof of these Propositions in the next section.

Our aim is now to use these propositions to construct  $\alpha(X, G)$  and a numerical invariant  $\mu(X, G)$ . The restriction map induces an injective homomorphism of  $\bar{k}$ -vector spaces

$$\mathrm{Ext}_{\bar{k}[G]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X)) \rightarrow \mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X)).$$

Since these vector spaces are of dimension 1 this is an isomorphism. We identify in what follows the class  $\beta(X, G)$  and the  $k$ -line  $L_1$  defined in Section 2 with their images in  $\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$ .

**Theorem 3.6.** *Let  $L_0$  be the  $k$ -line in  $\mathrm{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$  which is the image of  $1 \otimes \mathrm{Ext}_{k[C]}^{n+1}(W, k)$  (resp.  $\mathrm{Ext}_{k[C]}^{n+1}(W', k)$ ) under the isomorphism in (3.4) (resp. (3.5)) if  $n$  is odd (resp. if  $n$  is even). Let  $\alpha(X, G)$  be any generator of  $L_0$  over  $k$ , so that  $\alpha(X, G)$  is defined only up to multiplication by an element of  $k^*$ . Then*

$$(3.6) \quad \alpha(X, G) = \zeta \cdot \beta(X, G)$$

for an element  $\zeta \in \bar{k}^*$  which is well-defined up to multiplication by an element of  $k^*$ . The constant

$$(3.7) \quad \mu(X, G) = \zeta^{1-q} \in \bar{k}^*$$

lies in  $\bar{k}^*$  and is an invariant of the action of  $G$  on  $X$ .

*Proof.* This follows from Propositions 3.3, 3.4 and 3.5 and the fact that  $\text{Ext}_{\overline{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$  has dimension 1 over  $\overline{k}$ .  $\square$

**Corollary 3.7.** *Let  $F^{-1}$  be the anti-semilinear endomorphism of  $\text{Ext}_{\overline{k}[C]}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$  induced by the action of  $F$  on  $H^n(\mathcal{O}_X)$  and the action of  $F^{-1}$  on  $H^0(\mathcal{O}_X) = \overline{k}$ , described as in Theorem 2.9. Then  $F^{-1}$  acts on  $L_0$  by multiplication by  $\mu(X, G)$ . The constant  $\mu(X, G)$  lies in  $k^*$ .*

*Proof.* Let  $L_1$  be the  $k$ -line of  $\text{Ext}_{\overline{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X))$  introduced in Theorem 2.9. It follows from the definitions of Theorem 3.6 that

$$L_0 = \zeta L_1$$

for any  $\zeta$  satisfying (3.6). Suppose that  $c_1$  is a  $k$ -basis of  $L_1$  so that  $\zeta \cdot c_1$  is a  $k$ -basis of  $L_0$ . The endomorphism  $F^{-1}$  fixes  $L_1$  by Theorem 2.9. Hence since  $F^{-1}$  is anti-semilinear, we have

$$F^{-1}(\tau \cdot \zeta \cdot c_1) = \tau \cdot F^{-1}(\zeta) \cdot c_1 = \nu \cdot (\tau \cdot \zeta \cdot c_1)$$

for  $\tau \in k$ , where

$$\nu = \frac{F^{-1}(\zeta)}{\zeta}.$$

This proves that  $F^{-1}$  acts as multiplication by  $\nu$  on the  $k$ -line  $L_0$ , so  $\nu \in k^*$ . Hence

$$\nu = F(\nu) = \frac{\zeta}{F(\zeta)} = \zeta^{1-q} = \mu(X, G)$$

which completes the proof in view of (3.7).  $\square$

**Corollary 3.8.** *The action of  $F$  on  $\mathcal{O}_{Y_0}$  and on  $\mathcal{O}_X$  induces a  $k$ -linear action on  $W$  and  $W'$ . If  $n$  is odd (resp. even) then  $F$  acts on  $W$  (resp.  $W'$ ) by multiplication by  $\mu(X, G) \in k^*$ .*

*Proof.* The action of  $F$  on  $\mathcal{O}_{Y_0}$  and on  $\mathcal{O}_X$  is via the map  $\alpha \rightarrow \alpha^q$  on local sections, and is  $k$ -linear. Thus  $F$  respects the homomorphism  $\mathcal{O}_{Y_0} \rightarrow (\tilde{\pi})_* \mathcal{O}_X$  in Proposition 3.3, so it follows that  $F$  acts  $k$ -linearly on the one dimensional  $k$ -vector spaces  $W$  and  $W'$ . Suppose  $n$  is odd. Since  $C$  is cyclic there are isomorphisms

$$\begin{aligned} \text{Ext}_{k[C]}^{n+1}(W, k) &= H^{n+1}(C, \text{Hom}_k(W, k)) \\ (3.8) \quad &= \hat{H}^0(C, \text{Hom}_k(W, k)) \\ &= \text{Hom}_{k[C]}(W, k)/\text{Tr}_C \text{Hom}_k(W, k) \end{aligned}$$

Because  $W \cong k$  with trivial  $C$ -action, this gives an  $F^{-1}$ -equivariant isomorphism between  $\text{Ext}_{k[C]}^{n+1}(W, k)$  and  $\text{Hom}_k(W, k)$ . Recall that  $F^{-1}$  sends an element  $f \in \text{Hom}_k(W, k)$  to the homomorphism  $F^{-1}f$  defined by  $(F^{-1}f)(w) = F^{-1}(f(F(w)))$  for  $w \in W$ . Since

$\dim_k W = 1$  the action of  $F$  on  $W$  is given by multiplication by some  $\alpha \in k$ , and  $F$  fixed  $k$ . Hence

$$(F^{-1}f)(w) = f(\alpha w) = \alpha f(w)$$

so  $F^{-1}$  acts on  $\text{Hom}_k(W, k)$  by multiplication by  $\alpha$ . Thus  $\alpha$  is also the eigenvalue of  $F^{-1}$  on  $L_0 = \text{Ext}_{k[G]}^{n+1}(W, k)$ , so  $\alpha = \mu(X, G)$  by Corollary 3.8. The proof when  $n$  is even is similar.  $\square$

We now relate  $\mu(X, G)$  to zeta functions. Let  $\zeta(V/k, T)$  be the zeta function of a smooth projective variety  $V$  over  $k$ . Then  $\zeta(V/k, T) \in \mathbb{Z}_p[[T]]$ , and the congruence formula in [1, Exposé XXII, 3.1] is

$$(3.9) \quad \zeta(V/k, T) = \prod_{i=0}^{\dim(V)} \det(1 - FT|H^i(V, \mathcal{O}_V))^{(-1)^{i+1}} \pmod{p\mathbb{Z}_p[[T]]}.$$

Write this formula as

$$(3.10) \quad \zeta(V/k, T) \pmod{p\mathbb{Z}_p[[T]]} = \frac{\zeta_1(V/k, T)}{\zeta_0(V/k, T)}$$

where

$$(3.11) \quad \zeta_j(V/k, T) = \prod_{\substack{i \equiv j \pmod{2}}} \det(1 - FT|H^i(V, \mathcal{O}_V))$$

for  $j = 0, 1$ . Note that  $\zeta_1(V/k, T)$  and  $\zeta_0(V/k, T)$  could have a common zero, so the formula (3.10) does not imply that a zero of  $\zeta_1(V/k, T)$  (resp. of  $\zeta_0(V/k, T)$ ) is a zero (resp. pole) of  $\zeta(V/k, T) \pmod{p\mathbb{Z}_p[[T]]}$ .

**Corollary 3.9.** *If  $n$  is odd then  $\mu(X, G)^{-1}$  is a zero of  $\zeta_1(Y_0/k, T)$ . Suppose  $n$  is even. Let  $X_0$  be the quotient of  $X$  by the action of a lift  $\phi_X$  to  $\text{Gal}(X/Y_0)$  of the arithmetic Frobenius of  $\text{Gal}(Y/Y_0) \equiv \text{Gal}(\bar{k}/k)$ . Then  $X_0$  is a smooth projective variety over  $k$  such that  $X = \bar{k} \otimes_k X_0$ , and  $\mu(X, G)^{-1}$  is a zero of  $\zeta_0(X_0/k, T)$ .*

*Proof.* If  $n$  is odd, we have shown  $\mu(X, G)$  is the eigenvalue of  $F$  acting on the one-dimensional  $k$ -space  $W$  inside  $H^n(Y_0, \mathcal{O}_{Y_0})$ . Hence  $1 - \mu(X, G)^{-1}F$  is not invertible on  $H^n(Y_0, \mathcal{O}_{Y_0})$ , so  $\mu(X, G)^{-1}$  is a zero of  $\zeta_1(Y_0/k, T)$ . Suppose  $n$  is even. We have shown  $\mu(X, G)$  is the eigenvalue of  $F$  acting on a one-dimensional  $k$ -space  $W'$  which is a quotient of  $H^n(\mathcal{O}_X)^\Gamma$ , where  $\Gamma = \text{Gal}(X/Y_0)$ . It is shown in Lemma 4.1 below that  $\Gamma$  is the semidirect product of the normal subgroup  $G$  with the closure  $\overline{\langle \phi_X \rangle}$  of the subgroup generated by  $\phi_X$ . Since  $X \rightarrow Y_0$  is a pro-étale cover, it follows that  $X_0$  is smooth and projective over  $k$ , and that  $X = \bar{k} \otimes_k X_0$ . Hence  $H^n(\mathcal{O}_X) = \bar{k} \otimes_k H^n(X_0, \mathcal{O}_{X_0})$ , so  $H^n(\mathcal{O}_X)^{\overline{\langle \phi_X \rangle}} = H^n(X_0, \mathcal{O}_{X_0})$ . Thus  $W'$  is a subquotient of  $H^n(X_0, \mathcal{O}_{X_0})$ , so as above,  $1 - \mu(X, G)^{-1}F$  is not invertible on  $H^n(X_0, \mathcal{O}_{X_0})$ . Since we assumed  $n$  is even, this shows  $\mu(X, G)$  is a zero of  $\zeta_0(X_0/k, T)$ .  $\square$

**Example 3.10.** Suppose that  $X$  is a curve and  $n = 1$  in Hypothesis 3.1. Then

$$\zeta(Y_0, T) = \frac{P_1(Y_0, T)}{(1 - T)(1 - qT)}$$

when  $P_1(Y_0, T) \in \mathbb{Z}[T]$  is the characteristic polynomial of Frobenius acting on  $H_{et}^1(Y_0, \mathbb{Q}_\ell)$  for any prime  $\ell$  different from  $p$ . One has  $\zeta_0(Y_0, T) = (1 - T)$  since  $F$  acts trivially on  $H^0(Y_0, \mathcal{O}_{Y_0}) = k$  and  $H^j(Y_0, \mathcal{O}_{Y_0}) = 0$  if  $j > 1$ . Since  $(1 - qT)^{-1} \equiv 1 \pmod{p\mathbb{Z}_p[[T]]}$ , we conclude that

$$(3.12) \quad P_1(Y_0, T) \equiv \zeta_1(Y_0, T) \pmod{p\mathbb{Z}_p[[T]]}$$

Thus Corollary 3.9 shows  $\mu(X, G)^{-1}$  is a zero in  $k$  of  $P_1(Y_0, T)$ .

**Example 3.11.** Suppose that in example 3.10,  $X$  is an elliptic curve. Since  $\pi : X \rightarrow Y$  is an étale  $G = \mathbb{Z}/p$  cover,  $Y$  must be an ordinary elliptic curve, and  $Y_0$  has genus 1 over  $k$ . Because  $Y_0$  has a point defined over  $k$  by the Weil bound,  $Y_0$  is isomorphic to an elliptic curve over  $k$ , and  $Y_0$  is ordinary. Since  $\dim_k H^1(Y_0, \mathcal{O}_{Y_0}) = 1$ , we see that  $\mu(X, G)^{-1}$  is the unique zero of  $P_1(Y_0, T)$  in  $k$ . Thus  $\mu(X, G)$  is the image in  $k$  of the unit root of Frobenius acting on  $H_{et}^1(Y_0, \mathbb{Q}_\ell)$ . Suppose now that  $k = \mathbb{Z}/p$ , and let  $\underline{0}$  be the origin of  $Y_0$ , so that  $\underline{0}$  has residue field  $k$ . Let  $t$  be a uniformizing parameter in the local ring  $\mathcal{O}_{Y_0, \underline{0}}$ . There is a unique differential  $\omega \in H^0(Y_0, \Omega^1 Y_0/k)$  having an expansion

$$\omega = \sum_{\nu=1}^{\infty} c_\nu t^{\nu-1} dt$$

at  $\underline{0}$  for which  $c_1 = 1$ . In [3, Appendix 2, §5], Lang defines  $c_p$  to be the Hasse invariant of  $Y_0$ . Lang shows that changing  $t$  to  $bt$  for some  $b \in \mathcal{O}_{Y_0}^*$  changes  $c_p$  by  $\bar{b}^{p-1}$  where  $\bar{b}$  is the image of  $b$  in the residue field  $k$  of  $\mathcal{O}_{Y_0}$ . Since we have now assumed  $k = \mathbb{Z}/p$ , one has  $\bar{b}^{p-1} = 1$ , so  $c_p = c_p(Y_0) \in k$  is independent of the choice of  $t$ . The formula in [3, Appendix 2, §2, Thm. 2] shows  $c_p(Y_0)$  is the eigenvalue of  $F$  acting on  $H^1(Y_0, \mathcal{O}_{Y_0})$ . So we conclude from the fact that  $W = H^1(Y_0, \mathcal{O}_{Y_0})$  in Corollary 3.8 that  $\mu(X, G)$  is the Hasse invariant  $c_p(Y_0)$ .

#### 4. PROOF OF PROPOSITIONS 3.3, 3.4 AND 3.5.

Throughout this section we assume Hypothesis 3.1.

If  $\mathcal{F}$  is a  $G$ -sheaf on  $X$  (resp.  $Y$ ) we denote  $H^i(X, \mathcal{F})$  (resp.  $H^i(Y, \mathcal{F})$ ) by  $H^i(\mathcal{F})$ . The quotient morphism  $\pi : X \rightarrow Y = X/G$  induces a natural morphism  $\iota : \mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X)$  which identifies  $\mathcal{O}_Y$  with  $\pi_*(\mathcal{O}_X)^G$ . Moreover, we have a morphism  $\pi_*(\mathcal{O}_X) \rightarrow \mathcal{O}_Y$  induced by the trace element  $Tr_G$  of  $\bar{k}[G]$ . We denote by  $\iota_* : H^n(\mathcal{O}_Y) \rightarrow H^n(\mathcal{O}_X)^G$  and  $Tr_* : H^n(\mathcal{O}_X) \rightarrow H^n(\mathcal{O}_Y)$  the morphisms of  $\bar{k}[G]$ -modules respectively induced by

$\iota$  and  $Tr_G$ . We define  $L$  and  $L'$  by the exact sequence:

$$(4.1) \quad 0 \longrightarrow L \longrightarrow H^n(Y, \mathcal{O}_Y) \xrightarrow{\iota_*} H^n(\mathcal{O}_X)^G \longrightarrow L' \longrightarrow 0$$

#### 4.a. Proof of Proposition 3.3.

First we need some preliminary results.

**Lemma 4.1.** *The constant field of  $Y_0$  is  $k$ , and  $\text{Gal}(X/Y_0)$  is the semi direct product of  $\text{Gal}(Y/Y_0) \cong \text{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$  with the normal subgroup  $G = \text{Gal}(X/Y)$ . A lift  $\phi_X$  to  $\text{Gal}(X/Y_0)$  of the arithmetic Frobenius automorphism  $1 \otimes F$  of  $\text{Gal}(Y/Y_0)$  is well defined up to an element of  $G$ .*

*Proof.* The constant field of  $Y_0$  is  $k$  because  $Y = \bar{k} \otimes_k Y_0$  is a variety with constant field  $\bar{k}$ . The exact sequence

$$1 \rightarrow G \rightarrow \text{Gal}(X/Y_0) \rightarrow \text{Gal}(Y/Y_0) \rightarrow 1$$

splits because  $\text{Gal}(Y/Y_0)$  is pro-free on one generator. The rest of the Lemma is now clear.  $\square$

**Lemma 4.2.** *Recall that we have assumed that  $Y = \bar{k} \otimes_k Y_0$  for a smooth projective variety  $Y_0$  over  $k$  and that the induced morphism  $X \rightarrow Y_0$  is étale and Galois. Let  $\Gamma = \text{Gal}(X/Y_0)$ . The flat base change isomorphism  $H^n(Y, \mathcal{O}_Y) = \bar{k} \otimes_k H^n(Y_0, \mathcal{O}_{Y_0})$  gives an exact sequence*

$$(4.2) \quad 0 \longrightarrow W \longrightarrow H^n(Y_0, \mathcal{O}_{Y_0}) \xrightarrow{\iota_*} H^n(\mathcal{O}_X)^\Gamma \longrightarrow W' \longrightarrow 0$$

in which  $W$  and  $W'$  are  $k$ -vector spaces. The tensor product of (4.2) with  $\bar{k}$  over  $k$  is the exact sequence (4.1).

*Proof.* Recall that  $\phi_X \in \Gamma$  is a choice of lift of the arithmetic Frobenius  $\phi_Y \in \text{Gal}(Y/Y_0) \cong \text{Gal}(\bar{k}/k)$ . By Lemma 4.1,  $\Gamma$  is the semi-direct product of the normal subgroup  $G$  with the closure  $\overline{\langle \phi_X \rangle}$  of the subgroup generated by  $\phi_X$ . Hence  $\phi_X$  acts as a semi-linear automorphism of the  $\bar{k}$ -vector space  $H^n(\mathcal{O}_X)^G$ , and

$$(4.3) \quad (H^n(\mathcal{O}_X)^G)^{\overline{\langle \phi_X \rangle}} = H^n(\mathcal{O}_X)^\Gamma.$$

Lemma 2.6 shows

$$H^n(\mathcal{O}_X)^G = \bar{k} \otimes_k (H^n(\mathcal{O}_X)^G)^{\overline{\langle \phi_X \rangle}}.$$

since  $\phi_X$  is an automorphism of  $H^n(\mathcal{O}_X)^G$ . Combining this with (4.3) proves that

$$H^n(\mathcal{O}_X)^G = \bar{k} \otimes_k H^n(\mathcal{O}_X)^\Gamma.$$

Thus  $\iota_* : H^n(Y, \mathcal{O}_Y) \rightarrow H^n(\mathcal{O}_X)^\Gamma$  results from tensoring the natural homomorphism  $H^n(Y_0, \mathcal{O}_{Y_0}) \rightarrow H^n(\mathcal{O}_X)^\Gamma$  with  $\bar{k}$  over  $k$ . Since tensoring with  $\bar{k}$  over  $k$  is exact the tensor product of (4.2) with  $\bar{k}$  over  $k$  is (4.1) as required.  $\square$

Since it follows from Lemma 4.2 that

$$L = \bar{k} \otimes_k W \text{ and } L' = \bar{k} \otimes_k W'$$

we note that in order to complete the proof of Proposition 3.3 it suffices to show that  $L$  and  $L'$  are one-dimensional  $\bar{k}$ -vector spaces. This will be a consequence of Hypothesis 3.1 and the next proposition.

**Proposition 4.3.** *There exist  $\bar{k}$ -isomorphisms of vector spaces*

$$L \simeq H^n(G, \bar{k}), \quad L' \simeq H^{n+1}(G, \bar{k})$$

where  $G$  acts trivially on  $\bar{k}$ .

*Proof.* We start the proof by establishing a lemma.

**Lemma 4.4.** *The  $\bar{k}$ -linear map  $Tr_*$  induces an isomorphism of  $\bar{k}$ -vector spaces*

$$Tr_* : H^n(\mathcal{O}_X)_G \rightarrow H^n(\mathcal{O}_Y).$$

Moreover, the composition of  $Tr_*$  with  $\iota_*$  is the map

$$Tr_G : H^n(\mathcal{O}_X) \rightarrow H^n(\mathcal{O}_X)$$

induced by the multiplication by the trace element  $Tr_G$  of  $\bar{k}[G]$ .

*Proof.* The exact sequence of sheaves on  $Y$

$$\{0\} \rightarrow \mathcal{L} \rightarrow \pi_*(\mathcal{O}_X) \rightarrow \mathcal{O}_Y \rightarrow \{0\}$$

associated to  $Tr_G : \pi_*(\mathcal{O}_X) \rightarrow \mathcal{O}_Y$  induces a long exact sequence of  $\bar{k}$ -vector spaces

$$H^n(\mathcal{L}) \rightarrow H^n(\mathcal{O}_X) \rightarrow H^n(\mathcal{O}_Y) \rightarrow H^{n+1}(\mathcal{L}) \rightarrow \dots$$

Because  $\dim(Y) = n$ ,  $H^{n+1}(\mathcal{L}) = 0$ . Therefore  $Tr_* : H^n(\mathcal{O}_X) \rightarrow H^n(\mathcal{O}_Y)$  is onto with kernel  $M$  equal to the image of  $H^n(\mathcal{L})$  in  $H^n(\mathcal{O}_X)$ . To prove the Lemma, it will suffice to show that  $M$  is equal to the kernel  $I_G H^n(\mathcal{O}_X)$  of the surjection  $H^n(\mathcal{O}_X) \rightarrow H^n(\mathcal{O}_X)_G$ , where  $I_G$  is the augmentation ideal of  $\mathbb{Z}[G]$ .

Since  $Tr_* \circ (1-h) = 0$  for  $h \in G$ , we have  $I_G H^n(\mathcal{O}_X) \subset M$ . To show  $M \subset I_G H^n(\mathcal{O}_X)$ , we first observe that  $\mathcal{L} = I_G \pi_*(\mathcal{O}_X)$  since  $\pi_*(\mathcal{O}_X)$  is a locally free rank one  $\mathcal{O}_Y[G]$ -module. Let  $\mathcal{E}$  be the kernel of the natural morphism

$$\oplus_{h \in G} (1-h) : \oplus_{h \in G} \pi_*(\mathcal{O}_X) \mapsto I_G \pi_*(\mathcal{O}_X) = \mathcal{L}.$$

Since  $H^{n+1}(\mathcal{E}) = 0$ , this morphism induces a surjective map  $\tau : \oplus_{h \in G} H^n(\pi_*(\mathcal{O}_X)) \rightarrow H^n(\mathcal{L})$ . Therefore the image  $M$  of  $H^n(\mathcal{L})$  in  $H^n(\mathcal{O}_X)$  is contained in the image of the composition of  $\tau$  with the homomorphism  $H^n(\mathcal{L}) \rightarrow H^n(\mathcal{O}_X)$ . The latter composition is the map on cohomology induced by the homomorphism  $\oplus_{h \in G} (1-h) : \oplus_{h \in G} \pi_*(\mathcal{O}_X) \rightarrow$

$\pi_*(\mathcal{O}_X)$ . Since  $I_G$  is additively generated by  $1 - h$  as  $h$  ranges over  $G$ , this shows that  $M \subset I_G H^n(\mathcal{O}_X)$ , which completes the proof.  $\square$

We associate to the exact sequence

$$(4.4) \quad 0 \rightarrow H^0(\mathcal{O}_X) \rightarrow P_0 \rightarrow \cdots \rightarrow P_n \rightarrow H^n(\mathcal{O}_X) \rightarrow 0$$

the following commutative diagram:

$$\begin{array}{ccccccc} P_{n-1,G} & \xrightarrow{\partial} & P_{n,G} & \longrightarrow & H^n(\mathcal{O}_X)_G & \longrightarrow & 0 \\ Tr_G \downarrow & & Tr_G \downarrow & & Tr \downarrow & & \\ P_{n-1} & \xrightarrow{\partial} & P_n & \longrightarrow & H^n(\mathcal{O}_X) & \longrightarrow & 0 \end{array}$$

here  $Tr$  denotes the composite of the isomorphism  $Tr_* : H^n(\mathcal{O}_X)_G \rightarrow H^n(O_Y)$  and the identification  $\iota_* : H^n(O_Y) \rightarrow H^n(\mathcal{O}_X)^G \subset H^n(\mathcal{O}_X)$ ; thus

$$\begin{array}{ccccccc} 0 & \longrightarrow & \partial(P_{n-1,G}) & \longrightarrow & P_{n,G} & \longrightarrow & H^n(\mathcal{O}_X)_G \longrightarrow 0 \\ & & Tr_G \downarrow & & Tr_G \downarrow & & Tr \downarrow \\ 0 & \longrightarrow & \partial(P_{n-1}) & \longrightarrow & P_n & \longrightarrow & H^n(\mathcal{O}_X) \longrightarrow 0. \end{array}$$

Since the  $\bar{k}[G]$ -modules  $P_l$  are all projective, the map  $Tr_G$  induces an isomorphism from  $P_{l,G} \simeq P_l^G$ . Moreover, it follows from Lemma 4.4 that  $L$  identifies with the kernel of  $Tr_*$ . By using the Snake lemma we obtain an exact sequence:

$$(4.5) \quad 0 \rightarrow L \rightarrow \frac{\partial(P_{n-1})}{\partial(P_{n-1}^G)} \rightarrow \frac{P_n}{P_n^G} \rightarrow \frac{H^n(\mathcal{O}_X)}{Tr(H^n(\mathcal{O}_X))} \rightarrow 0$$

and hence an isomorphism of  $\bar{k}$ -vector spaces

$$(4.6) \quad L \simeq \frac{\partial(P_{n-1}) \cap P_n^G}{\partial(P_{n-1}^G)}.$$

Since the  $\bar{k}[G]$ -modules  $P_0, \dots, P_n$  are all projective, and hence injective, we can extend the complex

$$0 \rightarrow P_0 \rightarrow \dots \rightarrow P_n$$

to an injective resolution  $P^\bullet$  of  $H^0(\mathcal{O}_X) = \bar{k}$ . It follows from (4.6) that we have the following isomorphisms:

$$L \simeq H^n(\text{Hom}_{\bar{k}[G]}(\bar{k}, P^\bullet)) \simeq \text{Ext}_{\bar{k}[G]}^n(\bar{k}, \bar{k}) \simeq H^n(G, \bar{k}).$$

In view of Lemma 4.4 we can identify  $L'$  with the  $\bar{k}$ -vector space  $\frac{H^n(\mathcal{O}_X)^G}{Tr_G(H^n(\mathcal{O}_X))}$ . This leads us to the exact sequence

$$P_n^G \rightarrow \left( \frac{P_n}{\partial(P_{n-1})} \right)^G \rightarrow L' \rightarrow 0.$$

Using the complex  $P^\bullet$  once again, we deduce from the previous sequence that

$$(4.7) \quad L' \simeq \frac{\partial(P_n) \cap P_{n+1}^G}{\partial(P_n^G)} \simeq H^{n+1}(G, \bar{k}).$$

#### 4.b. Proof of Proposition 3.4.

We recall that  $C$  is a  $p$ -Sylow subgroup of  $G$ .

**Lemma 4.5.** *Define  $M(n)$  to be the  $\bar{k}[C]$ -module given by  $\bar{k}$  with trivial  $C$ -action if  $n$  is odd and by the quotient  $\bar{k}[C]/(\bar{k} \cdot \text{Tr}_C)$  if  $n$  is even, where  $\text{Tr}_C = \sum_{\sigma \in C} \sigma$  is the trace element of  $\bar{k}[C]$ . There is a  $\bar{k}[C]$ -module isomorphism*

$$H^n(\mathcal{O}_X) = M(n) \oplus M$$

in which  $M$  is a finitely generated free  $\bar{k}[C]$ -module.

*Proof.* Since  $C$  is a  $p$ -group and  $\bar{k}$  is of characteristic  $p$ , the ring  $\bar{k}[C]$  is local and artinian and hence every projective  $\bar{k}[C]$ -module is free and injective. The result now follows from the existence of the sequence (4.4) together with induction on  $n$ .  $\square$

Now we are using the notations of Proposition 3.3. We have an isomorphism of  $\bar{k}$ -vector spaces

$$\text{Ext}_{\bar{k}[C]}^{n+1}(F, \bar{k}) \simeq H^{n+1}(C, \text{Hom}_{\bar{k}}(F, \bar{k}))$$

for any  $\bar{k}[C]$ -module  $F$ , where  $\text{Hom}_{\bar{k}}(F, \bar{k})$  is endowed with the  $\bar{k}[C]$ -module structure given by

$$cf : m \rightarrow f(c^{-1}m) \quad \forall c \in C, \quad \forall f \in \text{Hom}_{\bar{k}}(F, \bar{k}).$$

Since  $C$  is cyclic and  $n$  is odd,

$$H^{n+1}(C, \text{Hom}_{\bar{k}}(F, \bar{k})) \simeq \text{Hom}_{\bar{k}}(F, \bar{k})^C / \text{Tr}_C(\text{Hom}_{\bar{k}}(F, \bar{k})).$$

This leads us to identify  $\text{Ext}_{\bar{k}[C]}^{n+1}(F, \bar{k})$  and  $\text{Hom}_{\bar{k}}(F, \bar{k})^C / \text{Tr}_C(\text{Hom}_{\bar{k}}(F, \bar{k}))$ . Under these identifications the map

$$\alpha : \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_Y), \bar{k}) \rightarrow \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k})$$

is induced by

$$\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_Y), \bar{k}) \rightarrow \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})^C$$

$$f \mapsto f \circ \text{Tr}_*$$

while the map

$$\beta : \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_Y), \bar{k}) \rightarrow \text{Ext}_{\bar{k}[C]}^{n+1}(\bar{k} \otimes_k W, \bar{k})$$

is the restriction map

$$\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_Y), \bar{k}) \rightarrow \text{Hom}_{\bar{k}}(L, \bar{k}).$$

Since, by hypothesis,  $L$  is a  $\bar{k}$ -sub-vector space of  $H^n(\mathcal{O}_Y)$  of dimension one, the  $\bar{k}$ -linear map  $\beta$  is clearly surjective and  $\text{Ker}(\beta)$  is a subvector space of  $\text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_Y), \bar{k})$  of codimension one.

Let  $f \in \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_Y), \bar{k})$  be an element of  $\text{Ker}(\alpha)$ . Then there exists  $h \in \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})$  such that  $f \circ Tr_* = Tr_C(h)$ . Let  $q$  be the index of  $C$  in  $G$  and let  $\{\tau \in S\}$  be a set of representatives of  $G/C$ . For any  $x \in H^n(\mathcal{O}_X)$  we have the equalities:

$$(4.8) \quad q(f \circ Tr_*(x)) = f \circ Tr_*(\sum_{\tau \in S} \tau x) = \sum_{c \in C} h(c \sum_{\tau \in S} (\tau x)) = h(Tr_G(x)).$$

It follows from Lemma 4.4 that any element  $a$  in  $L$  can be written  $a = Tr_*(x)$ , with  $Tr_G x = 0$ . Therefore we deduce from (4.8) that for every  $a \in L$

$$qf(a) = q(f \circ Tr_*(x)) = h(Tr_G x) = 0$$

and thus that  $f(a) = 0$ . We conclude that  $f \in \text{Ker}(\beta)$ . Hence we have proved that  $\text{Ker}(\alpha)$  is contained in  $\text{Ker}(\beta)$  and so that  $1 \leq \text{codim}(\text{Ker}(\alpha))$ . We now observe that to complete the proof of the proposition it suffices to prove that

$$\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})^C / Tr_C(\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k}))$$

is a  $\bar{k}$ -vector space of dimension one. As a consequence we will obtain that  $\text{codim}(\text{Ker}(\alpha)) \leq 1$  and we thereby deduce the equality  $\text{Ker}(\alpha)$  and  $\text{Ker}(\beta)$  and hence the surjectivity of  $\alpha$ .

Since  $n$  is odd we deduce from Lemma 4.5 that there exists a  $\bar{k}[C]$ -module isomorphism

$$H^n(\mathcal{O}_X) = \bar{k} \oplus M$$

where  $M$  is a finitely generated free  $\bar{k}[C]$ -module. The  $\bar{k}$ -vector space  $\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})$  splits into a direct sum

$$(4.9) \quad \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k}) = \text{Hom}_{\bar{k}}(\bar{k}, \bar{k}) \oplus \text{Hom}_{\bar{k}}(M, \bar{k}) = \bar{k} \oplus \text{Hom}_{\bar{k}}(M, \bar{k}).$$

Therefore

$$(4.10) \quad \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})^C = \bar{k} \oplus \text{Hom}_{\bar{k}}(M, \bar{k})^C$$

while

$$(4.11) \quad Tr_C(\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})) = \{0\} \oplus Tr_C(\text{Hom}_{\bar{k}}(M, \bar{k})).$$

Since  $M$  is a free  $\bar{k}[C]$ -module one easily checks that  $\text{Hom}_{\bar{k}}(M, \bar{k})$  is  $\bar{k}[C]$ -free and thus that  $\text{Hom}_{\bar{k}}(M, \bar{k})^C = Tr_C(\text{Hom}_{\bar{k}}(M, \bar{k}))$ . Hence we have proved that

$$\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})^C / Tr_C(\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})) \simeq \bar{k}$$

as required.  $\square$

**Remark.** One should note that the conclusions of Proposition 3.4 are incorrect when  $n$  is even.

#### 4.c. Proof of Proposition 3.5.

For the sake of simplicity in the proof we identify  $H^0(\mathcal{O}_X)$  and  $\bar{k}$  and  $W' \otimes_k \bar{k}$  with  $L'$ .

Since  $C$  is cyclic and  $n$  is even we have isomorphisms

$$(4.12) \quad \text{Ext}_{\bar{k}[C]}^{n+1}(L', \bar{k}) \simeq \hat{H}_{-1}(C, \text{Hom}_{\bar{k}}(L', \bar{k})) = \text{Hom}_{\bar{k}}(L', \bar{k})_{Tr_C}/(c-1)(\text{Hom}_{\bar{k}}(L', \bar{k})) = \text{Hom}_{\bar{k}}(L', \bar{k})$$

where, for a  $\bar{k}[C]$ -module  $Z$ , we let  $Z_{Tr_C}$  denote the submodule which is annihilated by  $Tr_C$ . By a similar argument we obtain the isomorphism

$$(4.13) \quad \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X)^G, \bar{k}) \simeq \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X)^G, \bar{k}).$$

Since  $L' = H^n(\mathcal{O}_X)^G/Tr_G(H^n(\mathcal{O}_X))$  is a  $\bar{k}$ -vector space of dimension one, it follows from (4.12) and (4.13) that  $\text{Ext}_{\bar{k}[C]}^{n+1}(L', \bar{k})$  is of dimension one and that the map

$$(4.14) \quad \text{Ext}_{\bar{k}[C]}^{n+1}(L', H^0(\mathcal{O}_X)) \rightarrow \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X)^G, H^0(\mathcal{O}_X))$$

can be identified with the homomorphism

$$\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X)^G/Tr_G(H^n(\mathcal{O}_X)), \bar{k}) \rightarrow \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X)^G, \bar{k})$$

obtained by composing an element of  $\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X)^G/Tr_G(H^n(\mathcal{O}_X)), \bar{k})$  with the natural surjection  $H^n(\mathcal{O}_X)^G \rightarrow H^n(\mathcal{O}_X)^G/Tr_G(H^n(\mathcal{O}_X))$ . This shows that (4.14) is an injective map which identifies  $\text{Ext}_{\bar{k}[C]}^{n+1}(L', \bar{k})$  with

$$(4.15) \quad \{f \in \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X)^G, \bar{k}) \text{ such that } f|Tr_G(H^n(\mathcal{O}_X)) = 0\}.$$

Since  $n$  is even, according to Lemma 4.5 we can decompose  $H^n(\mathcal{O}_X)$  into a direct sum of  $\bar{k}[C]$ -modules

$$H^n(\mathcal{O}_X) = M(n) \oplus M$$

where  $M(n) = \frac{\bar{k}[C]}{\bar{k}Tr_C}$  and  $M$  is a free  $\bar{k}[C]$ -module. This implies the following decompositions into direct sums of  $\bar{k}$ -vector spaces:

$$(4.16) \quad \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k}) = \text{Ext}_{\bar{k}[C]}^{n+1}(M(n), \bar{k}) \oplus \text{Ext}_{\bar{k}[C]}^{n+1}(M, \bar{k}) = \text{Ext}_{\bar{k}[C]}^{n+1}(M(n), \bar{k}) \oplus \{0\}$$

It now follows from (4.16) that

$$(4.17) \quad \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k}) = \text{Hom}_{\bar{k}}\left(\frac{\bar{k}[C]}{\bar{k}Tr_C}, \bar{k}\right)/(c-1)\text{Hom}_{\bar{k}}\left(\frac{\bar{k}[C]}{\bar{k}Tr_C}, \bar{k}\right).$$

The dimension over  $\bar{k}$  of the right-hand side of this equality is the dimension of the kernel of the multiplication by  $(c-1)$  on  $\text{Hom}_{\bar{k}}\left(\frac{\bar{k}[C]}{\bar{k}Tr_C}, \bar{k}\right)$ . The kernel of the multiplication by  $(c-1)$  naturally identifies with the vector space  $\text{Hom}_{\bar{k}}(M(n)_C, \bar{k})$ . One easily

checks that

$$(4.18) \quad M(n)_C \simeq \frac{\bar{k}[C]}{(c-1)\bar{k}C} \simeq \bar{k}Tr_C.$$

Hence we conclude that  $\text{Hom}_{\bar{k}}(M(n)_C, \bar{k})$  and thus  $\text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k})$  are of dimension one. We also have an isomorphism

$$(4.19) \quad \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k}) \simeq \hat{H}_{-1}(C, \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})) = \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})_{Tr_C}/(c-1)(\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})).$$

Therefore we deduce that the map

$$(4.20) \quad \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), H^0(\mathcal{O}_X)) \rightarrow \text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X)^G, H^0(\mathcal{O}_X))$$

is induced by the restriction homomorphism

$$(4.21) \quad \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})_{Tr_C} \rightarrow \text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X)^G, \bar{k}).$$

We note that for any  $x \in H^n(\mathcal{O}_X)$  there exists  $x'$  such that  $Tr_G(x) = Tr_C(x')$ . This shows that  $Tr_G(H^n(\mathcal{O}_X)) \subset H^n(\mathcal{O}_X)^G \cap Tr_C(H^n(\mathcal{O}_X))$ . Conversely, let  $\alpha = Tr_C(x)$  be an element of  $H^n(\mathcal{O}_X)^G \cap Tr_C(H^n(\mathcal{O}_X))$  and let  $\{g_i, 1 \leq i \leq q\}$  be a set of representatives of  $G/C$ . We have the equalities

$$q\alpha = \sum_{1 \leq i \leq q} g_i Tr_C(x) = Tr_G(x).$$

This shows that  $q\alpha$ , and hence also  $\alpha$ , belongs to  $Tr_G(H^n(\mathcal{O}_X))$ . We conclude that

$$(4.22) \quad Tr_G(H^n(\mathcal{O}_X)) = H^n(\mathcal{O}_X)^G \cap Tr_C(H^n(\mathcal{O}_X)).$$

It follows from (4.15) and (4.22) that the image of the map (4.21) is contained in the image of  $\text{Ext}_{\bar{k}[C]}^{n+1}(L', \bar{k})$ . Since  $\text{Ext}_{\bar{k}[C]}^{n+1}(H^n(\mathcal{O}_X), \bar{k})$  is of dimension one, in order to complete the proof of the proposition, it suffices to prove that the map (4.20) is not the zero map. Let  $f$  be a non-zero element of  $\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X)^G, \bar{k})$ , trivial on  $Tr_G(H^n(\mathcal{O}_X))$ , and let  $x$  be an element of  $H^n(\mathcal{O}_X)^G$  such that  $f(x) \neq 0$ . In view of (4.22) we know that  $x$  does not belong to  $Tr_C(H^n(\mathcal{O}_X))$ . Let  $V$  be a subvector space of  $H^n(\mathcal{O}_X)$ , containing  $Tr_C(H^n(\mathcal{O}_X))$ , and such that

$$H^n(\mathcal{O}_X) = V \oplus \bar{k}x.$$

Let  $g$  be the element of  $\text{Hom}_{\bar{k}}(H^n(\mathcal{O}_X), \bar{k})$  defined by  $g|V = 0$  and  $g(x) = f(x)$ . It follows from (4.22) that the restriction of  $g$  to  $Tr_G(H^n(\mathcal{O}_X))$  is trivial and therefore that  $g|H^n(\mathcal{O}_X)^G = f$ . This proves that, as required, (4.20) is not the zero map.  $\square$

## 5. EXAMPLES

Our goal is to provide examples of smooth projective varieties of arbitrary large dimension, defined over an algebraically closed field of characteristic  $p$ , endowed with the action of a cyclic group of order  $p$ , which fulfills Hypothesis 3.1.

Let  $p > 2$  be a prime and let  $\bar{k}$  be an algebraically closed field of characteristic  $p$ . Define  $G$  to be a cyclic group of order  $p$  with generator  $\sigma$ . We fix an action of  $G$  on the projective space  $\mathbb{P}_{\bar{k}}^{p-1}$  by letting  $\sigma$  send the point  $x = (x_0 : x_1 : x_2 : \dots : x_{p-1})$  to  $\sigma(x) = (x_1 : x_2 : x_1 : \dots : x_0)$ .

**Theorem 5.1.** *Let  $X$  be the closed subscheme of  $\mathbb{P}_{\bar{k}}^{p-1}$  defined by the homogeneous polynomial  $f_p = X_0 \dots X_{p-1} + X_0^{p-1}X_1 + X_1^{p-1}X_2 + \dots + X_{p-1}^{p-1}X_0$ . Then  $X$  is a smooth irreducible hypersurface of  $\mathbb{P}_{\bar{k}}^{p-1}$  of degree  $p$  on which  $G$  acts without fixed points. The quotient morphism  $X \rightarrow Y = X/G$  is étale, and  $Y$  is smooth, irreducible and projective of dimension  $n = \dim(X) = p - 2$ .*

*Proof.* Suppose that  $f_p$  is reducible. Let  $g$  be an irreducible, homogeneous divisor of  $f_p$  of degree  $m$  with  $m < p$ . We write  $f_p = gh$ . Since  $\sigma(f_p) = f_p$  then  $\sigma(g)$  divides  $f_p$  and thus either  $\sigma(g)$  divides  $g$  or  $\sigma(g)$  divides  $h$ . If  $\sigma(g)$  divides  $g$  there exists  $\lambda \in \bar{k}^*$  such that  $\sigma(g) = \lambda g$ . Since  $\sigma$  is of ordre  $p$  we deduce that  $\lambda^p = 1$  and so  $\lambda = 1$  and  $\sigma(g) = g$ . If  $\sigma(g) \neq g$  then  $\sigma(g)$  divides  $h$  and therefore  $g\sigma(g)\dots\sigma^{p-1}(g)$  divides  $f_p$ . Since  $\deg(f_p) = p$  the degree of  $g$  has to be 1. Therefore either  $\sigma(g) = g$  or  $g$  is homogeneous of degree 1 and  $f_p = g\sigma(g)\dots\sigma^{p-1}(g)$ . For any polynomial  $u$  we write  $Tr(u) = \sum_{0 \leq i \leq p-1} \sigma^i(u)$ . Since there is no monomial polynomial of degree  $< p$  invariant by sigma we conclude that if  $g$  is of degree  $< p$  and such that  $\sigma(g) = g$  there exists some polynomial  $u$  with  $g = Tr(u)$ . Since  $\bar{k}$  is of characteristic  $p$  then  $g(1) = Tr(u)(1, \dots, 1) = 0$  which is impossible since  $f_p(1, \dots, 1) = 1$ . We now suppose that  $f_p = g\sigma(g)\dots\sigma^{p-1}(g)$  with  $g = a_0X_0 + \dots + a_{p-1}X_{p-1}$ . Since  $f_p$  doesn't contain any monomial of the type  $X_i^p$  at least one of the  $a_i$ 's has to be equal to 0. Suppose for instance that  $a_0 = 0$ . Then the coefficient of  $X_0^{p-1}X_1$  in  $g\sigma(g)\dots\sigma^{p-1}(g)$  is equal to  $a_1^2a_2\dots a_{p-1}$ . This implies that  $a_i \neq 0$  for  $1 \leq i \leq p-1$ . Thus we will get in this product the monomial polynomials  $a_1a_2^2\dots a_{p-1}X_0^{p-1}X_2, \dots, a_1a_2\dots a_{p-1}^2X_0^{p-1}X_{p-1}$ . Since  $f_p$  doesn't contain such monomials we conclude that  $f_p$  can't be decomposed into such a product and thus that  $f_p$  is irreducible.

Now we claim that it doesn't exist  $x = (x_0, x_1, \dots, x_{p-1}) \neq (0, 0, \dots, 0)$  such that

$$(5.1) \quad f_p(x) = f'_{p,X_0}(x) = \dots f'_{p,X_{p-1}}(x) = 0.$$

Suppose  $x = (x_0, x_1, \dots, x_{p-1})$  satisfies (5.1). We have

$$(5.2) \quad f'_{p,X_i}(x) = x_0 \dots \hat{x}_i \dots x_{p-1} + x_{\sigma^{-1}(i)}^{p-1} - x_i^{p-2}x_{\sigma(i)} = 0, \quad 0 \leq i \leq p-1.$$

Using these equations one easily checks that if of the  $x'_i$ 's is equal to 0 then the others are. Let us denote by  $P$  the product  $x_0 \dots x_{p-1}$ . It follows from (5.2) that

$$(5.3) \quad x_i f'_{p,X_i}(x) = P + x_i x_{\sigma^{-1}(i)}^{p-1} - x_i^{p-1} x_{\sigma(i)} = 0, \quad 0 \leq i \leq p-1.$$

We deduce from (5.3) the equalities

$$(5.4) \quad x_i^{p-1} x_{i+1} = iP + x_0^{p-1} x_1, \quad 0 \leq i \leq p-1,$$

(by convention we set  $x_p = x_0$ ). Using (5.4) the equation

$$f_p(x) = P + \sum_{0 \leq i \leq p-1} x_i^{p-1} x_{i+1} = 0$$

can be written

$$(5.5) \quad f_p(x) = P + \frac{p(p-1)}{2}P + px_0^{p-1} x_1 = P = 0.$$

Then one (and thus all) of the  $x_i$ 's has to be equal to zero. This achieves the proof of the claim.

Since  $(1 : 1 : \dots : 1)$  is the unique fixed closed point of  $G$  acting on  $\mathbb{P}$  we conclude that as required  $X$  is a smooth irreducible hypersurface of  $\mathbb{P}_{\bar{k}}^{p-1}$  of degree  $p$  on which  $G$  acts without fixed points. Therefore  $\pi : X \rightarrow Y = X/G$  is a finite and étale morphism and  $Y$  is a smooth, irreducible and projective variety over  $\bar{k}$ .  $\square$

Since  $G$  is a cyclic  $p$ -group the  $\bar{k}$ -vector spaces  $H^n(G, \bar{k})$  and  $H^{n+1}(G, \bar{k})$  are of dimension 1 as seen before. Thus, to show that Hypothesis 3.1 is satisfied for  $(X, G)$ , it suffices to prove the following corollary.

**Corollary 5.2.** *The variety  $X$  is of dimension  $n$  and*

- i)  $H^j(\mathcal{O}_X) = \{0\}$  if and only if  $j \notin \{0, n\}$ .
- ii) If  $\bar{k}$  is an algebraic closure of  $\mathbb{Z}/p$  there is a model  $Y_0$  of  $Y$  over a finite field  $k_0$  such that the composition of  $X \rightarrow Y$  and  $Y \rightarrow Y_0$  is a Galois morphism  $X \rightarrow Y_0$ .

*Proof.* For simplicity, we will write  $\mathbb{P} = \mathbb{P}_{\bar{k}}^{p-1}$ . The ideal sheaf  $I_X$  of  $X$  is isomorphic to  $\mathcal{O}_{\mathbb{P}}(-p)$  since  $X$  is a hypersurface of degree  $p$ . We thus have an exact sequence

$$(5.6) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}}(-p) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow i_* \mathcal{O}_X \rightarrow 0$$

where  $i$  is the closed immersion  $X \rightarrow \mathbb{P}$ . Let  $S_m$  be the  $\bar{k}$ -vector space of homogeneous degree  $m$  polynomials in the homogeneous coordinate functions  $X_0, \dots, X_{p-1}$  (if  $m < 0$  we set  $S_m = 0$ ). From [4, Lemma 3.1] we have, for any  $m \in \mathbb{Z}$

$$H^j(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) = 0 \quad \text{if } j \neq 0, p-1$$

and

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) = S_m \quad \text{and} \quad H^{p-1}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-m-p))^{\vee}$$

where  $L^\vee = \text{Hom}(L, \bar{k})$  if  $L$  is a  $\bar{k}$ -vector space. Using this in the long exact cohomology sequence associated to (5.6) shows part (i), after noting, for any integer  $m$ , the equality

$$H^m(X, \mathcal{O}_X) = H^m(\mathbb{P}, i_*(\mathcal{O}_X)).$$

If  $\bar{k}$  is an algebraic closure of  $\mathbb{Z}/p$ , then there will be a model  $Y_1$  of  $Y$  over some finite extension  $k_1$  of  $\mathbb{Z}/p$ , and we can assume  $k_1$  is the field of constants of  $Y_1$ . Thus  $Y = \bar{k} \times_{k_1} Y_1$  and the Frobenius automorphism  $F_1$  of  $\bar{k}$  over  $k_1$  extends to a pro-generator of  $\text{Gal}(Y/Y_1) = \text{Gal}(\bar{k}/k_1)$ . Embed  $\bar{k}(X)$  into a separable closure  $\bar{k}(Y)^s$  of  $\bar{k}(Y)$ . Here  $\bar{k}(Y)^s$  is Galois over  $k_1(Y_1)$ , and a positive integral power  $F_1^m$  of  $F_1$  will lie in  $\text{Gal}(k(Y)^s/k(X))$ . We now let  $k_0$  be the fixed field of  $F_1^m$  acting on  $\bar{k}$  and we let  $Y_0 = k_0 \otimes_{k_1} Y_1$  to have part (ii) of the Corollary.  $\square$

We end this section with an other example of scheme  $X$  satisfying the properties in Theorem 5.1

**Proposition 5.3.** *Suppose  $X$  is an ordinary elliptic curve over  $\bar{k}$ . Translation by the order  $p > 2$  subgroup  $X[p](\bar{k})$  of  $\bar{k}$ -points of order  $p$  of  $X$  defines an étale action of the cyclic group  $G = \mathbb{Z}/p$  on  $X$  over  $\bar{k}$ . Define  $D$  to be the very ample degree  $p$  divisor on  $X$  which is the sum of the points in  $X[p](\bar{k})$ . There is a basis  $\{x_0, x_1, \dots, x_{p-1}\}$  over  $\bar{k}$  for  $H^0(X, \mathcal{O}_X(D))$  which is cyclically permuted by the action of a generator  $\sigma$  for  $G$ . Such a basis gives a  $G$ -equivariant closed immersion  $X \rightarrow \mathbb{P}_{\bar{k}}^{p-1}$  with  $G$  acting on  $\mathbb{P}_{\bar{k}}^{p-1}$  by cyclically permuting the homogeneous coordinates  $(x_0 : \dots : x_{p-1})$ . When  $p = 3$ , this defines  $X$  as a curve in  $\mathbb{P}_{\bar{k}}^2$  having the properties in Theorem 5.1.*

*Proof.* Since  $p \geq 3 = 2 \cdot \text{genus}(X) + 1$  we see that  $D$  is very ample by [2, Cor. IV.3.2]. Since the action of  $G$  on  $X$  is étale, the equivariant Euler characteristic  $[H^0(X, \mathcal{O}_X(D))] - [H^1(X, \mathcal{O}_X(D))]$  in  $G_0(\bar{k}[G])$  is in the image of the (injective) map  $K_0(\bar{k}[G]) \rightarrow G_0(\bar{k}[G])$ . Since  $H^1(X, \mathcal{O}_X(D)) = 0$  by Riemann Roch Theorem, we conclude that  $H^0(X, \mathcal{O}_X(D))$  is a projective  $kG$ -module, which must be a free module of rank one since  $\#G = p$  and  $\dim_{\bar{k}} H^0(X, \mathcal{O}_X(D)) = p$ . We now let  $x_0 \in H^0(X, \mathcal{O}_X(D))$  be a generator for  $H^0(X, \mathcal{O}_X(D))$  as a free  $\bar{k}G$ -module and we define  $x_i = \sigma^i x_0$  for  $0 \leq i \leq p-1$ . The Proposition is now clear.  $\square$

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